The Existence of Periodic Solutions to the Second-Order Discrete Equation

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Keywords: Periodic solution; Difference equations; Saddle point theorem

Abstract: This article is concerned with the existence of periodic solutions for nonlinear second-order difference equations. In this paper, the existence of periodic solution is obtained by using the critical point theorem and variational frameworks. First, we introduce some appropriate variational frameworks. The existence of periodic solutions is equivalent to the existence of critical points of the functional. Second, using critical point theorem, we obtain some critical points, the periodic solutions are obtained. The work replenishs a blank of this part.

1. Introduction

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ be the set of all natural numbers, integers and real numbers respectively. For $\mathrm{a}, \mathrm{b} \in \mathbb{Z}$, define $\mathbb{Z}(a)=\{a, a+1, a+2, \cdots\}$, when $a \leq b$, define $\mathbb{Z}(a,b)= \{a, a+1, a+2, \cdots, b\}$.

Consider the discrete system,

$$\Delta\left(p_n\left(\Delta x_{n-1}\right)^{\delta}\right)+f\left(n, x_n\right)=0, \tag{1}$$

where Δ is the forward difference operator defined by $\Delta x_n=x_{n+1}-x_n$, $\Delta^2 x_n=\Delta\left(\Delta x_n\right)$, and $(-1)^{\delta}=-1$, $\delta>0$, $f: \mathbb{Z} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous in the second variable , and $F(n, z)$ is defined as $F(n, z)=\int_0^z f(n, s) ds$, $p_{T+n}=p_n>0$, for some integer T . In this paper, we let $v_1=\min_{n \in \mathbb{Z}(1, T)}\left\{p_n\right\}>0$, $v_2=\max_{n \in \mathbb{Z}(1, T)}\left\{p_n\right\}>0$.

We may think of equation (1) as being a discrete analogue of a special case of the second order differential equation

$$\left(p(t) \varphi\left(u'\right)\right)'=f(t, u), \tag{2}$$

which has been studied by many authors [4, 7, 8]. In the case of $\varphi(u)=|u|^{\delta-2} u$, equation (2) has been discussed extensively in the literature, we may refer to [9, 10, 12, 13]. When $\delta=1$ and $f(n, u)=q_n u$, equation (1) has been investigated by many authors for results on oscillation, asymptotic behavior and boundary value problems [1, 2, 5, 6]. But the results on existence of periodic solutions of nonlinear difference equations are very scare in the literature [3, 14].

The main results are as follows:

Theorem 1.1. Suppose $F(n, z)$ satisfies

(A1) $F(n, z) \in C(\mathbb{R}, \mathbb{R})$ for each $n \in \mathbb{Z}$ and there exists a positive integer T such that for all $(n, z) \in \mathbb{Z} \times \mathbb{R}, F(n+T, z)=F(n, z)$.

(A2) There exist constants $R_1>0$ and $\alpha \in(1, \delta+1)$ such that for any $(n, z) \in \mathbb{Z} \times \mathbb{R}$, $|z| \geq R_1$

$$0 < z \cdot f(n,z) \le \alpha F(n,z), \tag{3}$$

(A3) There exist constants $a_1, a_2 > 0$ and $\gamma \in (1,\alpha]$ such that

$$F(n,z) \ge a_1 |z|^{\gamma} - a_2,\ \forall\ (n,z) \in \mathbb{Z} \times \mathbb{R}. \tag{4}$$

Then system (1) possesses at least one T-periodic solution.

Remark 1.1. By integrating (3), we have that

$$F(n,z) \le a_3 |z|^{\alpha} + a_4, \tag{5}$$

holds for some positive constants a_3 and a_4, which implies that $\lim_{|z|\to\infty} \frac{F(n,z)}{z^{\delta+1}} = 0$.

Theorem 1.2. Suppose that $F(n,z)$ satisfies (A1) and

(B1) there is a constant $M_0 > 0$ such that for all $(n,z) \in \mathbb{Z} \times \mathbb{R}$, $|f(n,z)| \le M_0$;

(B2) $F(n,z) \to +\infty$ uniformly for $n \in \mathbb{Z}$ as $|z| \to +\infty$.

Then system (1) possesses at least one T-periodic solution.

2. Some basic lemmas

In order to apply the critical point theory, we introduce some appropriate variational frameworks in this section.

Let S be the set of sequences $x = (\cdots, x_{-n}, \cdots, x_{-1}, x_0, x_1, \cdots, x_n, \cdots) = \{x_n\}_{n=-\infty}^{+\infty}$, i.e., $S = \{x = \{x_n\} \mid x_n \in \mathbb{R}, n \in \mathbb{Z}\}$. For any $x, y \in S$, $a, b \in \mathbb{R}$, $ax + by$ is defined by $ax + by := \{ax_n + by_n\}$, then S is a vector space.

For any given positive integer T, E_T is defined as a subspace of S by

$$E_T = \{x = \{x_n\} \in S : x_{n+T} = x_{n,} n \in \mathbb{Z}\}.$$

We note that E_T can be equipped with the inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$ as follows:

$$(x,y) = \sum_{j=1}^{T} x_j \cdot y_j, \forall x, y \in E_T, \tag{6}$$

$$\|x\| = \left(\sum_{j=1}^{T} x_j^2 \right)^{\frac{1}{2}}, \forall x \in E_T, \tag{7}$$

It is obvious that E_T with the inner product in the (6) is a finite-dimensional Hilbert space and linearly homeomorphic to R^T.

We define the functional J on E_T as follows:

$$J(x) = \frac{1}{\delta+1} \sum_{n=1}^{T} p_{n+1} (\Delta x_n)^{\delta+1} - \sum_{n=1}^{T} F(n, x_n). \tag{8}$$

It is easy to see that $J \in C^1(E_T, \mathbb{R})$ and for any $x \in E_T$, by using $x_T = x_0$, $x_{T+1} = x_1$ we can compute the partial derivative as

$$\frac{\partial J}{\partial x_n} = -\Delta\left[p_n (\Delta x_{n-1})^{\delta} \right] - f(n, x_n), n \in \mathbb{Z}(1,T)$$

then x is a critical point of J on E_T if and only if

$$\Delta\left(p_n\left(\Delta x_{n-1}\right)^{\delta}\right)+f\left(n,x_n\right)=0$$

By the periodicity of x_n and $f\left(n,z\right)$ in the first variable n, we have reduced the existence of periodic solutions of equation (1) to the existence of critical points of J on E_T . For convenience, we identify $x\in E_T$ with $x=\left(x_1,x_2,\cdots,x_T\right)^T$.

Denote $W=\left\{x\in E_T : x_i=v, v\in\mathbb{R}, i\in\mathbb{Z}\left(1,T\right)\right\}$ and $W^{\perp}=Y$, such that $E_T=W\oplus Y$. Denote the norm $\left\|\cdot\right\|_r$ on E_T as follows: $\left\|x\right\|_r=\left(\sum_{i=1}^{T}\left|x_i\right|^r\right)^{\frac{1}{r}}$, for all $x\in E_T$ and $r>1$. Clearly, $\left\|x\right\|=\left\|x\right\|_2$. Since || · ||r and || · || are equivalent, so it’s easy to get

$$T^{-1}\left\|x\right\|_r\le\left\|x\right\|\le T\left\|x\right\|_r, \forall x\in E_T, \tag{9}$$

Let X be a real Banach space, $I\in C^1\left(X,\mathbb{R}\right)$, that is, I is a continuously Frechet differentiable functional defined on X. The functional I is said to satisfy the Palais-Smale condition (P.S. condition for short) if any sequence $\left\{u_n\right\}\subset X$, for which $\left\{I\left(u_n\right)\right\}$ is bounded and when $n\to\infty$, $I'\left(u_n\right)\to 0$, then $\left\{u_n\right\}$ possesses a convergent subsequence in X.

Let B_r denote the open ball in X about 0 of radius r and let ∂B_r denote its boundary.

Lemma 2.1 (Saddle point theorem [11]) Let X be a real Banach space, $X=X_1\oplus X_2$ where $X_1\neq\left\{0\right\}$ and is finite dimensional. Suppose $I\in C^1\left(X,\mathbb{R}\right)$ satisfies the P-S condition and

(I1) there exist constants $\sigma>0$ and $\rho>0$ such that $I|_{\partial B_\rho\cap X_1}\le\sigma$;

(I2) there exist $e\in B_\rho\cap X_1$ and a constant $\omega>\sigma$ such that $I|_{e+X_1}\ge\omega$.

Then I possesses a critical value $c\ge\omega$ and $c=\inf_{h\in\Gamma}\max_{u\in B_\rho\cap X_1}I\left(h\left(u\right)\right)$, where

$$\Gamma=\left\{h\in C\left(\bar{B}_\rho\cap X_1,X\right): h|_{\partial\bar{B}_\rho\cap X_1}=id\right\}.$$

3. The proofs of main results

3.1 Proof of theorem 1.1

Firstly, we need to show that J satisfies the P.S. condition.

Clearly, $J\in C^1\left(E_T,\mathbb{R}\right)$. Let $x^{(k)}\in E_T, k\in\mathbb{Z}\left(1\right)$ be such that $\left\{J\left(x^{(k)}\right)\right\}$ is bounded and $J'\left(x^{(k)}\right)\to 0$ as $k\to\infty$. Then there exists a constant $M_1>0$ and $k_0\in\mathbb{Z}\left(1\right)$ such that $\left|J\left(x^{(k)}\right)\right|\le M_1$ for $k\in\mathbb{Z}\left(1\right)$ and $\left|\left\langle J'\left(x^{(k)}\right),x\right\rangle\right|\le\left\|x\right\|_2$ for $k\in\mathbb{Z}\left(k_0\right), x\in E_T$.

Let $h\left(x\right)=\frac{1}{\delta+1}\sum_{n=1}^{T}p_{n+1}\left(\Delta x_n\right)^{\delta+1}$, then,

$$h'\left(x\right)=\begin{pmatrix} p_1\left(\Delta x_T\right)^{\delta}-p_2\left(\Delta x_2\right)^{\delta} \\ p_2\left(\Delta x_1\right)^{\delta}-p_3\left(\Delta x_2\right)^{\delta} \\ \cdots \\ p_T\left(\Delta x_{T-1}\right)^{\delta}-p_1\left(\Delta x_T\right)^{\delta} \end{pmatrix},$$

By a simple computation, we can get

$$\left(h'(x),x\right)=(\delta+1)h(x)$$

Since

$$\left(J'\left(x^{(k)}\right),\ x^{(k)}\right)=\left(h'\left(x^{(k)}\right),\ x^{(k)}\right)-\sum_{n=1}^{T} f\left(n,x_n^{(k)}\right)\cdot x_n^{(k)}$$

we see that, for $k\in\mathbb{Z}(k_0)$

$$M_1+\frac{1}{\delta+1}\left\|x^{(k)}\right\|_2\ge\frac{1}{\delta+1}\left(J'\left(x^{(k)}\right),x^{(k)}\right)-J\left(x^{(k)}\right)$$

$$=\sum_{n=1}^{T}F\left(n,x_n^{(k)}\right)-\frac{1}{\delta+1}\sum_{n=1}^{T} f\left(n,x_n^{(k)}\right)\cdot x_n^{(k)}$$

For any $k\in\mathbb{Z}(k_0)$, denote

$$S_1^k=\left\{n\in\mathbb{Z}(1,T):\left|x_n^{(k)}\right|\ge R_1\right\}\ \text{ and }\ S_2^k=\left\{n\in\mathbb{Z}(1,T):\left|x_n^{(k)}\right|<R_1\right\}.$$

Then $S_1^k\cup S_2^k=\mathbb{Z}(1,T)$ and

$$M_1+\frac{1}{\delta+1}\left\|x^{(k)}\right\|_2\ge\sum_{n=1}^{T}F\left(n,x_n^{(k)}\right)-\frac{1}{\delta+1}\sum_{n=1}^{T} f\left(n,x_n^{(k)}\right)\cdot x_n^{(k)}$$

$$=\sum_{n=1}^{T}F\left(n,x_n^{(k)}\right)-\frac{1}{\delta+1}\sum_{n\in S_1^k} f\left(n,x_n^{(k)}\right)\cdot x_n^{(k)}\quad-\frac{1}{\delta+1}\sum_{n\in S_2^k} f\left(n,x_n^{(k)}\right)\cdot x_n^{(k)}.$$

In view of (3), we have

$$M_1+\frac{1}{\delta+1}\left\|x^{(k)}\right\|_2\ge\sum_{n=1}^{T}F\left(n,x_n^{(k)}\right)-\frac{\alpha}{\delta+1}\sum_{n\in S_1^k}F\left(n,x_n^{(k)}\right)-\frac{1}{\delta+1}\sum_{n\in S_2^k} f\left(n,x_n^{(k)}\right)\cdot x_n^{(k)}$$

$$=\left(1-\frac{\alpha}{\delta+1}\right)\sum_{n=1}^{T}F\left(n,x_n^{(k)}\right)+\frac{1}{\delta+1}\sum_{n\in S_2^k}\left[\alpha F\left(n,x_n^{(k)}\right)-f\left(n,x_n^{(k)}\right)\cdot x_n^{(k)}\right].$$

Since $\alpha F(n,z)-f(n,z)\cdot z$ is continuous with respect to $z\in\mathbb{R}$ for each $n\in\mathbb{Z}$, there exists a constant $M_2>0$ such that $\left|\alpha F(n,z)-f(n,z)\cdot z\right|\le M_2$, for all $z\in\mathbb{R}$ and $|z|\le R_1$, $n\in\mathbb{Z}(1,T)$. Thus

$$M_1+\frac{1}{\delta+1}\left\|x^{(k)}\right\|_2\ge\left(1-\frac{\alpha}{\delta+1}\right)\sum_{n=1}^{T}F\left(n,x_n^{(k)}\right)-\frac{TM_2}{\delta+1}.$$

By (4) and (9), we have

$$M_1+\frac{1}{\delta+1}\left\|x^{(k)}\right\|_2\ge\left(1-\frac{\alpha}{\delta+1}\right)a_1\sum_{n=1}^{T}\left|x_n^{(k)}\right|^{\gamma}-\left(1-\frac{\alpha}{\delta+1}\right)a_2T-\frac{1}{\delta+1}TM_2$$

$$\ge\left(1-\frac{\alpha}{\delta+1}\right)a_1\left(\frac{1}{T}\right)^{\gamma}\left\|x^{(k)}\right\|_2^{\gamma}-M_3$$

where $M_3=\left(1-\frac{\alpha}{\delta+1}\right)a_2T+\frac{1}{\delta+1}TM_2$. That is,

$$\left(1-\frac{\alpha}{\delta+1}\right)a_1\left(\frac{1}{T}\right)^{\gamma}\left\|x^{(k)}\right\|_2^{\gamma}-\frac{1}{\delta+1}\left\|x^{(k)}\right\|_2\le M_1+M_3$$

Because $\gamma\in(1,\alpha]$ and $0<\alpha<\delta+1$, we see that $\left\{\left\|x^{(k)}\right\|_2\right\}$ is bounded. Since E_T is finite dimensional, $\left\{x^{(k)}\right\}$ has a subsequence which is convergent in E_T. Therefore J satisfies the P.S. condition.

Now we prove that J satisfies (I1), (I2). Let $X_1=W$ and $X_2=Y$. Then for any $x\in X_2$,

$$\begin{aligned}J(x)&=\frac{1}{\delta+1}\sum_{n=1}^{T}p_{n+1}(\Delta x_n)^{\delta+1}-\sum_{n=1}^{T}F(n,x_n)\\&=\frac{1}{\delta+1}\sum_{n=1}^{T}p_{n+1}(x_{n+1}-x_n)^{\delta+1}-\sum_{n=1}^{T}F(n,x_n)\\&\ge\frac{v_1}{\delta+1}\left(\frac{1}{T}\right)^{\delta+1}\left[\sum_{n=1}^{T}(x_{n+1}-x_n)^2\right]^{\frac{1}{\delta+1}}-\sum_{n=1}^{T}\left(a_3|x_n|^{\alpha}+a_4\right)\\&=\frac{v_1}{\delta+1}\left(\frac{1}{T}\right)^{\delta+1}\left(x^TAx\right)^{\frac{1}{\delta+1}}-\sum_{n=1}^{T}\left(a_3|x_n|^{\alpha}+a_4\right),\end{aligned}$$

where $x=(x_1,x_2,\cdots,x_T)^T$ and

$$A=\begin{pmatrix}2&-1&0&\cdots&0&-1\\-1&2&-1&\cdots&0&0\\0&-1&2&\cdots&0&0\\\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\0&0&0&\cdots&2&-1\\-1&0&0&\cdots&-1&2\end{pmatrix}_{T\times T},$$

we get that $\lambda_1=0$ is an eigenvalue of A and $\xi=(v,v,\cdots,v)^T\in E_T$ is an eigenvector of A corresponding to 0, where $v\neq0$, $v\in\mathbb{R}$. Let $\lambda_2,\lambda_3,\cdots,\lambda_T$ be the other eigenvalues of A. By the matrix theory, we have $\lambda_j>0, j\in\mathbb{Z}(2,T)$. Without loss of generality, we may assume that $0<\lambda_2\le\lambda_3\le\cdots\le\lambda_T$, then for any $x\in Y$,

$$\begin{aligned}J(x)&\ge\frac{v_1}{\delta+1}\left(\frac{1}{T}\right)^{\delta+1}\left(\lambda_2\|x\|_2^2\right)^{\frac{\delta+1}{2}}-\sum_{n=1}^{T}\left(a_3|x_n|^{\alpha}+a_4\right)\\&\ge\frac{v_1}{\delta+1}\left(\frac{1}{T}\right)^{\delta+1}\lambda_2^{\frac{\delta+1}{2}}\|x\|_2^{\delta+1}-a_3\|x\|_{\alpha}^{\alpha}-a_4T\\&\ge\frac{v_1}{\delta+1}\left(\frac{1}{T}\right)^{\delta+1}\lambda_2^{\frac{\delta+1}{2}}\|x\|_2^{\delta+1}-a_3T^{\alpha}\|x\|_2^{\alpha}-a_4T.\end{aligned}$$

Because $\alpha<\delta+1$, then J is bounded from below. There exists a constant $-\omega>0$ such that $J|_Y\ge\omega$. Let $e=0$, so J satisfies (I2).

For any $x\in W$,

$$J(x) = -\sum_{n=1}^{T} F(n, x_n) \le -a_1 \sum_{n=1}^{T} |x_n|^{\gamma} + a_2 T = -a_1 \|x\|_{\gamma}^{\gamma} + a_2 T,$$

then $J(x) \to -\infty$, as $|x| \to \infty$. So there exists a constant ρ large enough such that $|x| \ge \rho$ and $J(x) < \omega - 1 =: \sigma$, so (I1) is satisfied. By saddle point theory, there exists at least one critical point.

3.2 Proof of theorem 1.2

Let J be defined as in (8). Clearly, $F \in C^1(E_T, \mathbb{R})$. In view of (B1), there exists a constant $M_4 > 0$ such that

$$|F(n,z)| \le M_4 + M_0 |z|, \forall (n,z) \in \mathbb{Z} \times \mathbb{R}. \tag{10}$$

We will first show that J satisfies the P.S. condition. In fact, suppose that $\{x^{(k)}\}$ is a sequence in E_T such that for any $k \in \mathbb{Z}(1), \left|J\left(x^{(k)}\right)\right| \le M_6$ for some positive constant M_5 and $\left|J'\left(x^{(k)}\right)\right| \to 0$ as $k \to \infty$, then for sufficiently large k, $\left|\left(J\left(x^{(k)}\right), x\right)\right| \le \|x\|_2$.

Let $x^{(k)} = y^{(k)} + w^{(k)}$, where $y^{(k)} \in Y, w^{(k)} \in W$. According to (8) and the periodicity of F, let $h(x) = \frac{1}{\delta+1} \sum_{n=1}^{T} p_{n+1} (\Delta x_n)^{\delta+1}$, we have

$$\left(J'\left(x^{(k)}\right), x\right) = \left(h'\left(x^{(k)}\right), x\right) - \sum_{n=1}^{T} f\left(n, x_n^{(k)}\right) \cdot x_n$$

Then for sufficiently large k,

$$\begin{aligned} \left|\left(h'\left(x^{(k)}\right), x\right)\right| &\le \sum_{n=1}^{T} \left| f\left(n, x_n^{(k)}\right) \cdot y_n^{(k)} \right| + \left\|y^{(k)}\right\|_2 \\ &\le M_0 \sum_{n=1}^{T} \left|y^{(k)}\right| + \left\|y^{(k)}\right\|_2 \\ &\le \left(M_0 \sqrt{T} + 1\right) \left\|y^{(k)}\right\|_2 \end{aligned}$$

and according to the proof of theorem (1), we get

$$\left|\left(h'\left(x^{(k)}\right), y^{(k)}\right)\right| = \left|\left(h'\left(y^{(k)}\right), y^{(k)}\right)\right| \ge \frac{v_1}{\delta+1} \left(\frac{1}{T}\right)^{\delta+1} \lambda_2^{\frac{\delta+1}{2}} \left\|y^{(k)}\right\|^{\delta+1}.$$

Thus we have $\frac{v_1}{\delta+1} \left(\frac{1}{T}\right)^{\delta+1} \lambda_2^{\frac{\delta+1}{2}} \left\|y^{(k)}\right\|_2^{\delta+1} \le \left(M_0 \sqrt{T} + 1\right) \left\|y^{(k)}\right\|_2$, which implies that $\left\{y_n^{(k)}\right\}$ is bounded. Next we need to prove that $\left\{w^{(k)}\right\}$ is bounded. In fact,

$$\begin{aligned} J\left(x^{(k)}\right) &= \frac{1}{\delta+1} \sum_{n=1}^{T} p_{n+1} \left(\Delta x_n^{(k)}\right)^{\delta+1} - \sum_{n=1}^{T} F\left(n, x_n^{(k)}\right) \\ &= \frac{1}{\delta+1} \sum_{n=1}^{T} p_{n+1} \left(\Delta y_n^{(k)}\right)^{\delta+1} - \sum_{n=1}^{T} F\left(n, w_n^{(k)}\right) + \sum_{n=1}^{T} \left[F\left(n, w_n^{(k)}\right) - F\left(n, x_n^{(k)}\right)\right]. \end{aligned}$$

So,

$$\left|\sum_{n=1}^{T}F\left(n,w_n^{(k)}\right)\right|\le\left|J\left(x^{(k)}\right)\right|+\frac{1}{\delta+1}\sum_{n=1}^{T}\left|p_{n+1}\left(\Delta y_n^{(k)}\right)^{\delta+1}\right|+\sum_{n=1}^{T}\left|F\left(n,w_n^{(k)}\right)-F\left(n,x_n^{(k)}\right)\right|$$

$$\le M_5+\frac{1}{\delta+1}\sum_{n=1}^{T}\left|p_{n+1}\left(\Delta y_n^{(k)}\right)^{\delta+1}\right|+\sum_{n=1}^{T}\left|f\left(n,w_n^{(k)}+\theta y_n^{(k)}\right)\right|\cdot\left|y_n^{(k)}\right|$$

$$\le M_5+\frac{1}{\delta+1}\sum_{n=1}^{T}\left|p_{n+1}\left(\Delta y_n^{(k)}\right)^{\delta+1}\right|+M_0\sqrt{T}\left\|y^{(k)}\right\|_2,$$

where $\theta\in(0,1)$. Since $\left\{y_n^{(k)}\right\}$ is bounded, this implies that $\left\{\sum_{n=1}^{T}F\left(n,w_n^{(k)}\right)\right\}$ is bounded.

By assumption (B2), we have that $\left\{w^{(k)}\right\}$ is bounded. If otherwise, there is no harm in assuming that $\left\|w^{(k)}\right\|_2\to\infty$ $\left\{w^{(k)}\right\}$ as $k\to\infty$. Since there exist $z^{(k)}\in\mathbb{R}$, $k\in\mathbb{Z}(1)$, such that $w^{(k)}=\left\{z^{(k)}\right\}\in E_T$,

then

$$\left\|w^{(k)}\right\|_2=\left(\sum_{n=1}^{T}\left|z_n^{(k)}\right|^2\right)^{\frac{1}{2}}=\sqrt{T}\left|z^{(k)}\right|\to\infty.$$

Since $F\left(n,w_n^{(k)}\right)=F\left(n,z^{(k)}\right)$, we have $F\left(n,w_n^{(k)}\right)\to\infty$ as $k\to\infty$. This contradicts the fact that $\left\{\sum_{n=1}^{T}F\left(n,w_n^{(k)}\right)\right\}$ is bounded, so P.S. condition is satisfied.

Now we will check that the conditions in the Saddle Point theorem hold. For any $y\in Y$,

$$J(y)=\frac{1}{\delta+1}\sum_{n=1}^{T}p_{n+1}\left(\Delta y_n\right)^{\delta+1}-\sum_{n=1}^{T}F\left(n,y_n\right)$$

$$\ge L\|y\|_2^{\delta+1}-\sum_{n=1}^{T}\left(M_0\left|y_n\right|^{\alpha}+M_4\right)$$

$$\ge L\|y\|_2^{\delta+1}-M_0\sum_{n=1}^{T}\left|y_n\right|-M_4T$$

$$\ge L\|y\|_2^{\delta+1}-M_0\sqrt{T}\|y\|-M_4T,$$

where $L=\frac{v_1}{\delta+1}\left(\frac{1}{T}\right)^{\delta+1}\lambda_2^{\frac{\delta+1}{2}}$. It is easy to know that $J(y)$ is bounded from below, that is, $J(y)\ge M_6$, for all $y\in Y$ for some constant M_6. Thus take $\omega=M_6$, then we have $J(y)\ge\omega,\forall y\in Y$. Let $e=0$, then (I2) holds.

For any $w\in W$, then $w=\{z\}$, we have

$$J(w)=\frac{1}{\delta+1}\sum_{n=1}^{T}p_{n+1}\left(\Delta w_n\right)^{\delta+1}-\sum_{n=1}^{T}F(n,z)=-\sum_{n=1}^{T}F(n,z).$$

According to condition (B2), this implies that $J(w)\to\infty$ as $\|w\|\to\infty$. Let $\sigma:=\omega-1$, then there exists a sufficiently large $\rho>0$ such that $J(w)\le\sigma,\forall w\in W$, and $\|w\|=\rho$. Thus condition (I1) is satisfied. By the Saddle Point theorem, the proof of theorem 1.2 is complete.

References

[1] Agarwal R.P., Difference Equations and Inequalities: Theory, Methods and Applications, Marcel Dekker, New York, 2000.

[2] Ahlbrandt C.D. and Peterson A.C., Discrete Hamitonian Systems: Difference Equation, Continued Fractions, and Riccati Equation, Kluwer Academic Publishers, (1996).

[3] Cai X.C., Yu J.S. and Guo Z.M., Periodic Solutions of a Class of Nonliear Difference Equations via Critical Point Method, Comput. Math. Appl. 52 (2006) 1639-1647.

[4] Castro A. and Shivaji R., Nonnegetive Solutions to a Semilinear Dirivhlet Problem in a Ball are Positive ans Radially Symmetric, Comm. PDE. 14(8-9) (1989) (1091-1100).

[5] Chen S. and Erbe L.H., Riccati Techniques and Discrete Oscillations, J. Math. Anal. Appl. 142 (1989) 468-487.

[6] Chen S. and Erbe L.H., Oscillation and Nonoscillation for Systems of Self-adjoint Second-order Difference Equations, SIAM J. Math. Anal. 20 (1989) 939-949.

[7] Esteban J.R. and Vazguez J.L., On the Equation of Turbulent Filtration in One-dimensional Porous Media, Nonlinear Analysis 10(12)(1986)1303-1325.

[8] Herrero M.A. and Vazguez J.L., On the propagation properties of a nonliear degenerate parobolic equation, Comm Partial Differential Equations 7(12)(1982),1381-1402.

[9] Li W. T., Oscillation of certain Second-order Nonliear Differential Equations, J. Math. Anal. Appl. 217 (1998) 1-14.

[10] Marini M. and Zezza P., On the Asympototic Behavior of the Solutions of a Class of Second Order Linear Differential Equations, J. Diff. Equat. 28 (1978) 1-17.

[11] Rabinowitz P.H., Minimax Methods in Critical Point Theory with Application to Differential Equation, CBMS Reg. Conf. Ser. Math., vol.65, Amer. Math. Soc., Providence, RI, 1986.

[12] Wen L. Z., Asypototic and Oscillation of Second-order Functional Differential Equation, Science in China 2, (1986) 147-161, (in chinese).

[13] Wong P.J.Y. and Agarwal R.P., Oscillatory Behavior of Solutions certain Second Order Nonlinear Differencial Equations, J. Math. Anal. Appl. 198 (1996) 113-354.

[14] Zhou Z., Yu J.S. and Guo Z.M., The Existence of Periodic and Subharmonic Solutions to Subquadratic Discrete Hamiltonian Systems, ANZIAM J. 47 (2005) 89-102.